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Decomposition of Representations of CAR Induced by Bogoliubov Endomorphisms

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Abstract

In a Fock representation, a non-surjective Bogoliubov transformation of CAR leads to a reducible representation. For the case that the corresponding Bogoliubov operator has finite corank, the decomposition into irreducible subrepresentations is clarified. In particular, it turns out that the number of appearing subrepresentations is completely determined by the corank.

1 Introduction

Unitary i.e. surjective Bogoliubov operators U correspond to Bogoliubov automorphisms ϱ_U of the canonical anticommutation relations (CAR). In a Fock representation π_P, Bogoliubov automorphisms lead to irreducible representations. More precisely, the representation $\pi_P \circ \varrho_U$ is again a Fock representation. The unitary equivalence class of $\pi_P \circ \varrho_U$ is in a certain way classifiable by the operator U^*PU [1, 2, 3].

Here we study Bogoliubov endomorphisms ϱ_V, where V is a non-surjective Bogoliubov operator. Throughout this paper, we consider a separable, infinite dimensional Hilbert space $\mathcal{K}$ with complex conjugation Γ, and we use Araki's formalism of the selfdual CAR algebra $\mathcal{C}(\mathcal{K},\Gamma)$ which is equivalent to the more familiar notion of Clifford algebras over real Hilbert spaces [2, 5]. We suppose that the corank of the Bogoliubov operator is finite. Since V is non-surjective, a representation of the form $\pi_P \circ \varrho_V$ is reducible. It turns out that the decomposition into irreducibles is determined by the corank of V. If it is an even number, say $2N$, we prove that $\pi_P \circ \varrho_V$ decomposes into 2^N mutually equivalent Fock representations. The explicit form of those subrepresentations can be seen from our proofs. On the other hand, if the corank of V is an odd number, say $2N+1$, we find that $\pi_P \circ \varrho_V$ decomposes into 2^{N+1} irreducibles where we have two equivalence classes of 2^N mutually equivalent subrepresentations each. Furthermore, we investigate what happens when those representations become restricted to the even subalgebra $\mathcal{C}(\mathcal{K},\Gamma)^+$ of $\mathcal{C}(\mathcal{K},\Gamma)$, in both cases. It turns out that the situation then becomes inverted somehow.

2 Preliminaries

As the underlying test function space, we consider a separable, infinite dimensional Hilbert space $\mathcal{K}$ with an antiunitary involution Γ (complex conjugation), $\Gamma^2 = \mathbf{1}$, fulfilling

$$\langle \Gamma f, \Gamma g \rangle = \langle g, f \rangle, \qquad f, g \in \mathcal{K}.$$

Let $\mathcal{C}_0(\mathcal{K}, \Gamma)$ be the unital $*$-algebra which is algebraically generated by the range of the linear embedding $B : \mathcal{K} \to \mathcal{C}_0(\mathcal{K}, \Gamma)$, where the following relations hold,

$$B(f)^* = B(\Gamma f), \qquad \{B(f)^*, B(g)\} = \langle f, g \rangle \mathbf{1}.$$

There is a unique C^*-norm of $\mathcal{C}_0(\mathcal{K}, \Gamma)$ which satisfies

$$\|B(f)\| = \frac{1}{\sqrt{2}} \sqrt{\|f\|^2 + \sqrt{\|f\|^4 - |\langle f, \Gamma f \rangle|^2}}.$$

The C^*-completion is referred as the selfdual CAR algebra over $(\mathcal{K}, \Gamma)$ and is denoted by $\mathcal{C}(\mathcal{K}, \Gamma)$. Elements of the set

$$\mathcal{I}(\mathcal{K}, \Gamma) = \{V \in \mathcal{B}(\mathcal{K}) \,|\, [V, \Gamma] = 0,\ V^* V = \mathbf{1}\}$$

of Γ commuting isometries on $\mathcal{K}$ are called Bogoliubov operators. Each Bogoliubov operator $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ induces a unital $*$-endomorphism ϱ_V of $\mathcal{C}(\mathcal{K}, \Gamma)$, defined by its action on generators,

$$\varrho_V(B(f)) = B(Vf),$$

ϱ_V is precisely an automorphism if V is surjective, i.e. $\ker V^* = \{0\}$. In this paper, we consider Bogoliubov operators with $\dim \ker V^* < \infty$ which are Fredholm operators; we set

$$M_V = \dim \ker V^* = \dim \operatorname{coker} V = \operatorname{index} V^* = -\operatorname{index} V$$

and define subsets of $\mathcal{I}(\mathcal{K}, \Gamma)$,

$$\mathcal{I}^n(\mathcal{K}, \Gamma) = \{V \in \mathcal{I}(\mathcal{K}, \Gamma) \mid M_V = n\}.$$

Now we come to the states of $\mathcal{C}(\mathcal{K}, \Gamma)$ we are interested in.

Definition 2.1 *A state ω of $\mathcal{C}(\mathcal{K}, \Gamma)$ is called quasifree, if for all $n \in \mathbf{N}$*

$$\omega(B(f_1) \cdots B(f_{2n+1})) = 0, \tag{1}$$

$$\omega(B(f_1) \cdots B(f_{2n})) = (-1)^{\frac{n(n-1)}{2}} \sum_\sigma \operatorname{sign}\sigma \prod_{j=1}^n \omega(B(f_{\sigma(j)}) B(f_{\sigma(n+j)})) \tag{2}$$

holds. The sum runs over all permutations $\sigma \in \mathcal{S}_{2n}$ with the property

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \qquad \sigma(j) < \sigma(j+n), \qquad j = 1, \ldots, n. \tag{3}$$

Quasifree states are therefore completely characterized by their two point functions. Moreover, it is known that there is a one to one correspondence between the set of quasifree states and the set

$$Q(\mathcal{K}, \Gamma) = \{S \in \mathcal{B}(\mathcal{K}) \,|\, S = S^*,\ 0 \le S \le \mathbf{1},\ S + \Gamma S \Gamma = \mathbf{1}\},$$

given by the formula

$$\omega(B(f)^* B(g)) = \langle f, Sg \rangle. \tag{4}$$

The quasifree state characterized by Eq. (4) will be referred as ω_S. A quasifree state, composed with a Bogoliubov endomorphism is again a quasifree state, namely we have $\omega_S \circ \varrho_V = \omega_{V^* S V}$. The projections in $Q(\mathcal{K}, \Gamma)$ are called basis projections. For a basis projection P, the state ω_P is pure and is called a Fock state. The corresponding GNS representation $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ is irreducible, it is called the Fock representation. The vector $|\Omega_P\rangle \in \mathcal{H}_P$ is called the Fock vacuum and the representation space $\mathcal{H}_P$ is precisely the antisymmetric Fock space over $P\mathcal{K}$. Araki proved [1] the following

Lemma 2.2 *Let P be a basis projection and let ω be a state of $\mathcal{C}(\mathcal{K},\Gamma)$ which satisfies*

$$\omega(B(f)B(f)^*) = 0, \qquad f \in P\mathcal{K}. \tag{5}$$

Then ω is the Fock state $\omega = \omega_P$.

Powers and Størmer [3] developed an important criterion for the quasiequivalence of representations being induced by gauge invariant quasifree states. Araki [1] generalized this criterion for arbitrary quasifree states.

Theorem 2.3 *Two quasifree states ω_{S_1} and ω_{S_2} of $\mathcal{C}(\mathcal{K},\Gamma)$ induce quasiequivalent representations, if and only if the difference $S_1^{\frac{1}{2}} - S_2^{\frac{1}{2}}$ is Hilbert Schmidt class.*

In this paper, we study representations of the form $\pi_P \circ \varrho_V$. If V is surjective, i.e. $V \in \mathcal{I}^0(\mathcal{K},\Gamma)$, then ϱ_V is an automorphism, V^*PV again a basis projection and $\pi_P \circ \varrho_V$ again an (irreducible) Fock representation, $\pi_P \circ \varrho_V = \pi_{V^*PV}$. But if $V \in \mathcal{I}^n(\mathcal{K},\Gamma)$, $n > 0$, then we will see that $\pi_P \circ \varrho_V$ is always reducible. We are interested in the decomposition into its irreducible subrepresentations and their structure. The first step in this investigation is the following Lemma 2.4 which, in the frame of common CAR formalism, was proven by Rideau [4], and, in the frame of the selfdual CAR formalism, has been formulated by Binnenhei [5, 6]. We define

$$N_V = \dim(\ker V^* \cap P\mathcal{K}),$$

$0 \leq N_V \leq \frac{1}{2}M_V$, and we choose an orthonormal basis (ONB) $\{k_n, n = 1, 2, \ldots, N_V\}$ of the space $\ker V^* \cap P\mathcal{K}$. Further we introduce the set I_{N_V} of multi-indices

$$I_{N_V} = \{\beta = (\beta_1, \beta_2, \ldots, \beta_l) \,|\, 1 \leq l \leq N_V,\ \beta_1, \ldots, \beta_l \in \mathbf{N},\ 1 \leq \beta_1 < \beta_2 < \cdots < \beta_l \leq N_V\} \cup \{0\}.$$

We remark that I_{N_V} consists of 2^{N_V} elements. Next we set $a_j = T_P(-1)\pi_P(B(k_j))$, $j = 1, 2, \ldots, N_V$, where $T_P(-1)$ denotes the unitary operator in $\mathcal{B}(\mathcal{H}_P)$, unique up to a phase, which implements the automorphism α_{-1} of $\mathcal{C}(\mathcal{K},\Gamma)$, defined by its action on generators,

$$\alpha_{-1}(B(f)) = -B(f) \tag{6}$$

in the Fock representation π_P. The implementing $T_P(-1)$ exists because α_{-1} leaves any quasifree state invariant [1]. We define for $\beta \in I_{N_V}$

$$A_\beta = a_{\beta_1} a_{\beta_2} \cdots a_{\beta_l}, \qquad \beta \neq 0, \qquad A_0 = \mathbf{1}; \qquad |\Omega_\beta\rangle = A_\beta |\Omega_P\rangle.$$

By $\mathcal{H}_\beta$ we denote the subspaces of $\mathcal{H}_P$ being generated by the action of $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K},\Gamma))$ on vectors $|\Omega_\beta\rangle$, and by $\pi^{(\beta)}_{V^*PV}$ the restrictions of $\pi_P \circ \varrho_V$ to $\mathcal{H}_\beta$, $\beta \in I_{N_V}$.

Lemma 2.4 *Let ω_P be a Fock state, $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ the corresponding Fock representation and ϱ_V a Bogoliubov endomorphism of $\mathcal{C}(\mathcal{K},\Gamma)$, $V \in \mathcal{I}(\mathcal{K},\Gamma)$. Then the representation $\pi_P \circ \varrho_V$ decomposes into 2^{N_V} cyclic subrepresentations,*

$$\pi_P \circ \varrho_V = \bigoplus_{\beta \in I_{N_V}} \pi^{(\beta)}_{V^*PV} \tag{7}$$

where $(\mathcal{H}_\beta, \pi^{(\beta)}_{V^*PV}, |\Omega_\beta\rangle)$ are GNS representations of the state $\omega_P \circ \varrho_V = \omega_{V^*PV}$.

However, this decomposition is a decomposition into cyclic representations, but in general not into irreducibles because V^*PV is in general not a basis projection. In our analysis of the decomposition of representations $\pi_P \circ \varrho_V$, the number M_V turns out to be the important quantity. It is useful to distinguish the even case, $M_V = 2N$, and the odd case, $M_V = 2N + 1$.

3 The Even Case: $M_V = 2N$

Let us begin our investigation with the special case that $V \in \mathcal{I}^2(\mathcal{K}, \Gamma)$, i.e. $M_V = 2$. Then we choose an ONB $\{e_+, e_-\}$ of $\ker V^*$ with the property $e_+ = \Gamma e_-$. This is always possible because $\ker V^*$ is Γ-invariant since $[V, \Gamma] = 0$. Following Araki [1, 2], in a Γ-invariant space it is always possible to choose a Γ-invariant ONB, say here f_1 and f_2, $\Gamma f_j = f_j$, $j = 1, 2$. Then vectors

$$e_\pm = \frac{1}{\sqrt{2}}(f_1 \pm i f_2)$$

have the required property. Now we have a look at the numbers $\lambda_+, \lambda_- \in [0, 1]$,

$$\lambda_\pm = \langle e_\pm, P e_\pm \rangle.$$

Since $P + \Gamma P \Gamma = \mathbf{1}$ we have $\lambda_+ + \lambda_- = 1$,

$$\lambda_+ + \lambda_- = \langle e_+, P e_+ \rangle + \langle e_-, P e_- \rangle = \langle e_+, P e_+ \rangle + \langle \Gamma e_+, P \Gamma e_+ \rangle = \langle (P + \Gamma P \Gamma) e_+, e_+ \rangle = 1.$$

Suppose first that one of these numbers equals zero, say $\lambda_- = 0$, and therefore $\lambda_+ = 1$. We then have the case $e_+ \in P\mathcal{K}$ and $e_- \in (\mathbf{1} - P)\mathcal{K}$. The state $\omega_P \circ \varrho_V = \omega_{V^* P V}$ is then pure (a Fock state) because $V^* P V$ is a basis projection; namely we find

$$\begin{aligned} (V^* P V)^2 &= V^* P V V^* P V = V^* P (\mathbf{1} - |e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|) P V \\ &= V^* P V - V^* |e_+\rangle\langle e_+| V = V^* P V. \end{aligned}$$

But by Lemma 2.4 it follows that the representation $\pi_P \circ \varrho_V$ decomposes into two equivalent irreducible (Fock) representations,

$$\pi_P \circ \varrho_V = \pi^{(1)}_{V^* P V} \oplus \pi^{(2)}_{V^* P V}.$$

The corresponding representation spaces $\mathcal{H}^{(1)}_{V^* P V}$ and $\mathcal{H}^{(2)}_{V^* P V}$ are generated by the action of $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))$ on vectors $|\Omega^{(1)}\rangle$ and $|\Omega^{(2)}\rangle$, respectively,

$$|\Omega^{(1)}\rangle = |\Omega_P\rangle, \qquad |\Omega^{(2)}\rangle = a_+ |\Omega_P\rangle, \qquad a_+ = T_P(-1) \pi_P(B(e_+)). \tag{8}$$

Now suppose the case that both numbers $\lambda_\pm \neq 0$ and therefore $\lambda_\pm \neq 1$. We argue that in this case $N_V = 0$, i.e. $\ker V^* \cap P\mathcal{K} = \{0\}$: Suppose there is a $v \in \ker V^* \cap P\mathcal{K}$. Since $v \in \ker V^*$ we have to write

$$v = \mu_+ e_+ + \mu_- e_-,$$

with complex numbers $\mu_\pm \in \mathbf{C}$. On the other hand, since $v \in P\mathcal{K}$ it follows $\Gamma v \in (\mathbf{1} - P)\mathcal{K}$ and therefore

$$\langle \Gamma v, P \Gamma v \rangle = 0.$$

We have $\Gamma v = \bar{\mu}_+ e_- + \bar{\mu}_- e_+$, and since $\langle e_\pm, P e_\mp \rangle = 0$ by

$$\langle e_+, P e_- \rangle = \langle e_+, (\mathbf{1} - \Gamma P \Gamma) e_- \rangle = -\langle e_+, \Gamma P e_+ \rangle = -\langle P e_+, \Gamma e_+ \rangle = -\langle e_+, P e_- \rangle$$

this reads

$$\langle \Gamma v, P \Gamma v \rangle = |\mu_+|^2 \langle e_-, P e_- \rangle + |\mu_-|^2 \langle e_+, P e_+ \rangle = |\mu_+|^2 \lambda_- + |\mu_-|^2 \lambda_+ = 0.$$

But since $\lambda_\pm$ both are non-zero and positive by assumption this establishes $\mu_+ = \mu_- = 0$ and therefore $v = 0$. By Lemma 2.4 we obtain that in this case $(\mathcal{H}_P, \pi_P \circ \varrho_V, |\Omega_P\rangle)$ is a GNS representation of the state $\omega_P \circ \varrho_V$. However, as we will see, in this case the state is a mixture [7]. In the following we use the fact that if a state ω is a mixture of two pure states ω_1 and ω_2 of a C^*-algebra, $\omega_1 \neq \omega_2$,

$$\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2, \qquad \lambda_1, \lambda_2 > 0, \qquad \lambda_1 + \lambda_2 = 1,$$

with associated GNS representations $(\mathcal{H}_j, \pi_j, |\Omega_j\rangle)$, $j = 1, 2$, a GNS representation for ω is given by the direct sum

$$(\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2, \sqrt{\lambda_1}|\Omega_1\rangle \oplus \sqrt{\lambda_2}|\Omega_2\rangle).$$

Lemma 3.1 *Let ω_P be a Fock state and let $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ be the corresponding Fock representation of $\mathcal{C}(\mathcal{K}, \Gamma)$. Let $V \in \mathcal{I}^2(\mathcal{K}, \Gamma)$ be a Bogoliubov operator and choose an ONB $\{e_+, e_-\}$ of $\ker V^*$ with the property that $e_+ = \Gamma e_-$. If both numbers*

$$\lambda_\pm = \langle e_\pm, P e_\pm\rangle \neq 0, \tag{9}$$

then the state $\omega_P \circ \varrho_V$ is a mixture of two Fock states $\omega_{P_\pm}$,

$$\omega_P \circ \varrho_V = \lambda_+ \omega_{P_+} + \lambda_- \omega_{P_-} \tag{10}$$

where the basis projections P_+ and P_- are explicitely given by

$$P_\pm = V^* P V + \lambda_\pm^{-1} V^* P (E_\mp - E_\pm) P V, \qquad E_\pm = |e_\pm\rangle\langle e_\pm|. \tag{11}$$

Moreover, the representation $\pi_P \circ \varrho_V$ is cyclic and decomposes therefore into a direct sum of two irreducible representations which are equivalent.

Proof. Since the orthonormal vectors e_+, e_- span the kernel of V^* (i.e. the cokernel of V), generators $B(e_+)$ and $B(e_-)$ anticommute with each generator $B(Vf)$, $f \in \mathcal{K}$. Therefore operators

$$a_\pm = T_P(-1)\pi_P(B(e_\pm))$$

lie in the commutant of $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))$,

$$a_\pm \in \pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))'.$$

Since $\lambda_\pm \neq 0$ we have the well defined, normed vectors in $\mathcal{H}_P$,

$$|\Omega_\pm\rangle = \lambda_\pm^{-\frac{1}{2}} a_\pm |\Omega_P\rangle. \tag{12}$$

We define states ω_+, ω_- of $\mathcal{C}(\mathcal{K}, \Gamma)$ by

$$\omega_\pm(x) = \langle \Omega_\pm | \pi_P \circ \varrho_V(x) | \Omega_\pm\rangle = \lambda_\pm^{-1} \langle \Omega_P | \pi_P \circ \varrho_V(x) \pi_P(B(e_\mp)B(e_\pm)) | \Omega_P\rangle, \qquad x \in \mathcal{C}(\mathcal{K}, \Gamma),$$

such that we find

$$\omega_P \circ \varrho_V = \lambda_+ \omega_+ + \lambda_- \omega_-$$

by $\{B(e_+), B(e_-)\} = \mathbf{1}$. We are able to compute the two point functions of ω_+ and ω_- by reading the permutation formulae (2), (3) for the quasifree state ω_P,

$$\begin{aligned}
\omega_\pm(B(f)B(g)) &= \lambda_\pm^{-1} \langle \Omega_P | \pi_P \circ \varrho_V(B(f)B(g)) \pi_P(B(e_\mp)B(e_\pm)) | \Omega_P\rangle \\
&= \lambda_\pm^{-1} \omega_P(B(Vf)B(Vg)B(e_\mp)B(e_\pm)) \\
&= \lambda_\pm^{-1} \omega_P(B(Vf)B(Vg)) \omega_P(B(e_\mp)B(e_\pm)) \\
&\qquad + \lambda_\pm^{-1} \omega_P(B(Vf)B(e_\pm)) \omega_P(B(Vg)B(e_\mp)) \\
&\qquad - \lambda_\pm^{-1} \omega_P(B(Vf)B(e_\mp)) \omega_P(B(Vg)B(e_\pm)) \\
&= \langle \Gamma f, V^* P V g\rangle + \lambda_\pm^{-1} \langle \Gamma f, V^* P e_\pm\rangle \langle \Gamma g, V^* P e_\mp\rangle \\
&\qquad - \lambda_\pm^{-1} \langle \Gamma f, V^* P e_\mp\rangle \langle \Gamma g, V^* P e_\pm\rangle.
\end{aligned}$$

Since $V^* e_\pm = 0$ and $[V, \Gamma] = 0$ we find

$$\langle \Gamma g, V^* P e_\pm \rangle = \langle V^* \Gamma P e_\pm, g \rangle = \langle V^* (\Gamma - P\Gamma) e_\pm, g \rangle = -\langle V^* P e_\mp, g \rangle = -\langle e_\mp, PVg \rangle.$$

Hence we can write

$$\omega_\pm(B(f)B(g)) = \langle \Gamma f, P_\pm g \rangle,$$

where

$$P_\pm = V^* PV + \lambda_\pm^{-1} V^* P(E_\mp - E_\pm)PV, \qquad E_\pm = |e_\pm\rangle\langle e_\pm|.$$

Using $V^* E_\pm = 0 = E_\pm V$ and $\Gamma E_\pm = E_\mp \Gamma$, one finds easily the relation

$$P_\pm + \Gamma P_\pm \Gamma = \mathbf{1},$$

namely we compute

$$\begin{aligned} P_\pm + \Gamma P_\pm \Gamma &= V^* PV + \lambda_\pm^{-1} V^* P(E_\mp - E_\pm)PV + V^*(\mathbf{1} - P)V \\ &\qquad + \lambda_\pm^{-1} V^* (\mathbf{1} - P)\Gamma(E_\mp - E_\pm)\Gamma(\mathbf{1} - P)V \\ &= \mathbf{1} + \lambda_\pm^{-1} V^* P(E_\mp - E_\pm)PV + \lambda_\pm^{-1} V^* P(E_\pm - E_\mp)PV = \mathbf{1}. \end{aligned}$$

In the next step we show that $P_\pm^2 = P_\pm$ i.e. that P_+ and P_- are basis projections. For simplicity we check at first only the case $P_+^2 = P_+$. We begin with some helpful formulae. Since $E_+ + E_-$ is the projection onto the kernel of V^* we have

$$VV^* = \mathbf{1} - E_- - E_+. \tag{13}$$

By $\lambda_\pm = \langle e_\pm, P e_\pm \rangle$ we get

$$E_\pm P E_\pm = \lambda_\pm E_\pm, \tag{14}$$

and since $\langle e_\pm, P e_\mp \rangle = 0$ we find

$$E_+ P E_- = E_- P E_+ = 0. \tag{15}$$

Define

$$P_{+,1} = V^* PV, \qquad P_{+,2} = -\lambda_+^{-1} V^* P E_+ PV, \qquad P_{+,3} = \lambda_+^{-1} V^* P E_- PV$$

such that

$$P_+ = P_{+,1} + P_{+,2} + P_{+,3}.$$

We obtain the following list of products by using Eqs. (13), (14) and (15).

$$\begin{aligned} P_{+,1}^2 &= V^* PV - V^* P E_+ PV - V^* P E_- PV, \\ P_{+,1} P_{+,2} &= (1 - \lambda_+^{-1}) V^* P E_+ PV, \\ P_{+,1} P_{+,3} &= \lambda_+^{-1} (1 - \lambda_-) V^* P E_- PV, \\ P_{+,2} P_{+,1} &= (1 - \lambda_+^{-1}) V^* P E_+ PV, \\ P_{+,2}^2 &= (\lambda_+^{-1} - 1) V^* P E_+ PV, \\ P_{+,2} P_{+,3} &= 0, \\ P_{+,3} P_{+,1} &= \lambda_+^{-1} (1 - \lambda_-) V^* P E_- PV, \\ P_{+,3} P_{+,2} &= 0, \\ P_{+,3}^2 &= \lambda_+^{-2} \lambda_- (1 - \lambda_-) V^* P E_- PV. \end{aligned}$$

By using only $\lambda_+ + \lambda_- = 1$ we compute

$$\begin{aligned} P_+^2 &= \sum_{k,l=1}^{3} P_{+,k} P_{+,l} \\ &= V^*PV + (-1+1-\lambda_+^{-1}+1-\lambda_+^{-1}+\lambda_+^{-1}-1)V^*PE_+PV \\ &\qquad +(-1+1+1+\lambda_+^{-1}\lambda_-)V^*PE_-PV \\ &= V^*PV - \lambda_+^{-1}V^*PE_+PV + \lambda_+^{-1}V^*PE_-PV \\ &= P_+. \end{aligned}$$

By interchanging all $+$ and $-$ indices this reads $P_-^2 = P_-$. We have proven that P_+ and P_- are both basis projections, hence the states ω_+ and ω_- satisfy

$$\omega_\pm(B(f)B(f)^*) = 0, \qquad f \in P_\pm \mathcal{K},$$

and therefore they are Fock states $\omega_\pm = \omega_{P_\pm}$ by Lemma 2.2. As already mentioned, $(\mathcal{H}_P, \pi_P \circ \varrho_V, |\Omega_P\rangle)$ is a GNS representation of the state $\omega_P \circ \varrho_V$ by Lemma 2.4. According to the decomposition (10) of this state, its GNS representation therefore splits into two Fock representations. Finally we emphasize that these Fock representations are equivalent: The difference

$$P_+ - P_- = \lambda_+^{-1}\lambda_-^{-1}V^*P(E_- - E_+)PV$$

is obviously Hilbert Schmidt class because E_- and E_+ are rank-one-projections. Using Theorem 2.3 one finds that ω_{P_+} and ω_{P_-} give rise to equivalent representations, q.e.d.

We observe the somewhat amazing phenomenon that in the case that one of the numbers λ_+ and λ_- vanishes, the state $\omega_P \circ \varrho_V$ remains pure but the representation $(\mathcal{H}_P, \pi_P \circ \varrho_V, |\Omega_P\rangle)$ is no longer cyclic; it splits into two equivalent irreducibles, and, on the other hand, if $\lambda_\pm \neq 0$ both, then the representation $(\mathcal{H}_P, \pi_P \circ \varrho_V, |\Omega_P\rangle)$ remains cyclic but the state $\omega_P \circ \varrho_V$ becomes a mixture of two pure states. In both cases we find that $\pi_P \circ \varrho_V$ decomposes into two equivalent irreducible (Fock) representations; this fact is true for each basis projection P and each $V \in \mathcal{I}^2(\mathcal{K}, \Gamma)$ and will be used for proving the main result of this section.

Theorem 3.2 *Let ω_P be a Fock state and let $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ be the corresponding Fock representation of the selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$. Further let $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ be a Bogoliubov operator with finite even corank, i.e. $M_V = 2N$, $N \in \mathbf{N}_0$, and let ϱ_V be the corresponding Bogoliubov endomorphism. Then the representation $\pi_P \circ \varrho_V$ decomposes into 2^N mutually equivalent irreducible (Fock) representations.*

Proof. We choose a Γ-invariant ONB $\{e_n, n \in \mathbf{N}\}$ of $\mathcal{K}$, i.e. $e_n = \Gamma e_n$, $n \in \mathbf{N}$. Further we choose a Γ-invariant ONB $\{f_n, n = 1, 2, \ldots, 2N\}$ of $\ker V^*$, $f_n = \Gamma f_n$, $n = 1, 2, \ldots, 2N$. Moreover, we define

$$f_{2N+n} = V e_n, \qquad n \in \mathbf{N}.$$

Since $\mathcal{K}$ is the direct sum of the range of V and the kernel of V^*, the set $\{f_n, n \in \mathbf{N}\}$ forms another Γ-invariant ONB of $\mathcal{K}$ and we can write

$$V = \sum_{n=1}^{\infty} |f_{2N+n}\rangle\langle e_n|.$$

We define Bogoliubov operators, the unitary $V_0 \in \mathcal{I}^0(\mathcal{K}, \Gamma)$, and $V_2 \in \mathcal{I}^2(\mathcal{K}, \Gamma)$ by

$$V_0 = \sum_{n=1}^{\infty} |f_n\rangle\langle e_n|, \qquad V_2 = \sum_{n=1}^{\infty} |e_{n+2}\rangle\langle e_n|$$

such that

$$V = V_0 V_2^N, \qquad \varrho_V = \varrho_{V_0} \varrho_{V_2}^N.$$

Since V_0 is unitary $P_0 = V_0^* P V_0$ is again a basis projection, and

$$\pi_P \circ \varrho_V = \pi_{P_0} \circ \varrho_{V_2}^N.$$

Now we can use the foregoing results iteratively. Since π_{P_0} is a Fock representation $\pi_{P_0} \circ \varrho_{V_2}$ decomposes into two equivalent Fock representations, say

$$\pi_{P_0} \circ \varrho_{V_2} = \pi_{P_1^{(1)}} \oplus \pi_{P_2^{(1)}}$$

where, using Eq. (11), in any case $P_j^{(1)} - V_2^* P_0 V_2$ is Hilbert Schmidt class, $j = 1, 2$. In the next step, we find a decomposition into four equivalent Fock representations,

$$\pi_{P_0} \circ \varrho_{V_2}^2 = \pi_{P_1^{(1)}} \circ \varrho_{V_2} \oplus \pi_{P_2^{(1)}} \circ \varrho_{V_2} = \pi_{P_1^{(2)}} \oplus \pi_{P_2^{(2)}} \oplus \pi_{P_3^{(2)}} \oplus \pi_{P_4^{(2)}}$$

where in any case $P_j^{(2)} - (V_2^*)^2 P_0 V_2^2$ is Hilbert Schmidt class, $j = 1, 2, 3, 4$, and so on. At the end one finds

$$\pi_P \circ \varrho_V = \pi_{P_0} \circ \varrho_{V_2}^N = \bigoplus_{j=1}^{2^N} \pi_{P_j^{(N)}}$$

where $P_j^{(N)} - (V_2^*)^N P_0 V_2^N = P_j^{(N)} - V^* P V$ is Hilbert Schmidt class, $j = 1, 2, \ldots, N$. Using Theorem 2.3, we obtain that all representations $\pi_{P_j^{(N)}}$ must be mutually equivalent. The proof is complete, q.e.d.

Because of the decomposition of $\pi_P \circ \varrho_V$ into 2^N irreducibles, there must exist a set of 2^N disjoint projections in $\mathcal{B}(\mathcal{H}_P)$ (the projections onto invariant subspaces), commuting with $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))$, which sum up to unity. To complete the picture, we construct such a set. By setting

$$g_{\pm j} = \frac{1}{\sqrt{2}}(f_j \pm i f_{N+j}), \qquad j = 1, 2, \ldots, N,$$

we find an ONB $\{g_j, j = \pm 1, \pm 2, \ldots, \pm N\}$ of $\ker V^*$ with the property $g_j = \Gamma g_{-j}$, $j = 1, 2, \ldots N$. We define the mutually commuting projections

$$\Pi_j^{\pm} = \pi_P(B(g_{\pm j})^* B(g_{\pm j})) \in \pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))'$$

such that $\Pi_j^+ + \Pi_j^- = \mathbf{1}$, $j = 1, 2, \ldots, N$. Then we have 2^N projections of the form

$$\Pi_{\epsilon_1, \epsilon_2, \ldots, \epsilon_N} = \Pi_1^{\epsilon_1} \Pi_2^{\epsilon_2} \cdots \Pi_N^{\epsilon_N}, \qquad \epsilon_j = \pm, \qquad j = 1, 2, \ldots, N$$

which have the desired properties.

4 The odd case: $M_V = 2N + 1$

Following Araki [1], a projection $F \in \mathcal{B}(\mathcal{K})$ with the property that $F \perp \Gamma F \Gamma$ and

$$(\mathbf{1} - F - \Gamma F \Gamma)\mathcal{K} = \text{span}\{f_0\},$$

where $f_0 \in \mathcal{K}$ is a normed, Γ-invariant vector is called a partial basis projection with Γ-codimension 1. By $(\mathcal{H}_F, \pi_F, |\Omega_F\rangle)$ we denote the Fock representation of $\mathcal{C}((F + \Gamma F \Gamma)\mathcal{K}, \Gamma)$ corresponding to F.

(Note that F is a basis projection of $((F+\Gamma F\Gamma)\mathcal{K},\Gamma)$.) It is proven in [1] that for such an F there exists an irreducible representation $\pi_{(F,f_0)}$ of $\mathcal{C}(\mathcal{K},\Gamma)$ on the Fock space $\mathcal{H}_F$, uniquely determined by

$$\pi_{(F,f_0)}(B(f)) = \frac{1}{\sqrt{2}}\langle f_0, f\rangle T_F(-1) + \pi_F(B(Ff+\Gamma F\Gamma f)). \tag{16}$$

Here $T_F(-1)$ denotes the unitary operator which implements the automorphism $\alpha_{-1}(B(f)) = -B(f)$ of $\mathcal{C}((F+\Gamma F\Gamma)\mathcal{K},\Gamma)$ in π_F. To the representation $\pi_{(F,f_0)}$, there corresponds the non-quasifree state of $\mathcal{C}(\mathcal{K},\Gamma)$,

$$\omega_{(F,f_0)}(x) = \langle\Omega_F|\pi_{(F,f_0)}(x)|\Omega_F\rangle, \qquad x\in\mathcal{C}(\mathcal{K},\Gamma).$$

Araki proved the following

Lemma 4.1 *Let F be a partial basis projection with Γ-codimension 1 and define $S\in Q(\mathcal{K},\Gamma)$ by*

$$S = \frac{1}{2}(\mathbf{1}+F-\Gamma F\Gamma). \tag{17}$$

Then the quasifree state ω_S of $\mathcal{C}(\mathcal{K},\Gamma)$ decomposes according to

$$\omega_S = \frac{1}{2}(\omega_{(F,f_0)}+\omega_{(F,-f_0)}). \tag{18}$$

Pure states $\omega_{(F,f_0)}$ and $\omega_{(F,-f_0)}$ give rise to inequivalent representations.

Now suppose a given Bogoliubov operator $W\in\mathcal{I}^1(\mathcal{K},\Gamma)$ and an arbitrary basis projection $P\in\mathcal{B}(\mathcal{K})$. We claim that $S=W^*PW$ is always of the form (17) so that Lemma 4.1 can be applied.

Lemma 4.2 *Let P be a basis projection of $(\mathcal{K},\Gamma)$ and $W\in\mathcal{I}^1(\mathcal{K},\Gamma)$. Then there exists a partial basis projection F with Γ-codimension 1 such that*

$$S = W^*PW = \frac{1}{2}(\mathbf{1}+F-\Gamma F\Gamma). \tag{19}$$

By Lemma 4.1 the state $\omega_S=\omega_P\circ\varrho_W$ splits into two pure states. Furthermore, the representation $\pi_P\circ\varrho_W$ decomposes into two inequivalent representations.

Proof. Let g_0 be the Γ-invariant, normed vector which spans $\ker W^*$, and we introduce $G_0=|g_0\rangle\langle g_0|$. We then define

$$E = 4(S-S^2) = 4(W^*PW - W^*PWW^*PW) = 4W^*P(\mathbf{1}-WW^*)PW = 4WPG_0PW.$$

Since g_0 is Γ-invariant we find

$$\langle g_0, Pg_0\rangle = \langle\Gamma g_0, P\Gamma g_0\rangle = \langle(\mathbf{1}-P)g_0, g_0\rangle = 1-\langle g_0, Pg_0\rangle$$

and therefore

$$\langle g_0, Pg_0\rangle = \frac{1}{2}, \qquad G_0PG_0 = \frac{1}{2}G_0.$$

This leads us to the fact that E is a projection,

$$E^2 = 16W^*PG_0PWW^*PG_0PW = 16W^*PG_0P(\mathbf{1}-G_0)PG_0PW = 2E-E = E.$$

On the other hand we find $\Gamma E\Gamma = E$,

$$\Gamma E\Gamma = 4W^*\Gamma PG_0P\Gamma W = 4W^*(\mathbf{1}-P)G_0(\mathbf{1}-P)W = 4W^*PG_0PW = E$$

since $W^*G_0 = 0 = G_0W$ (remember that G_0 is the projection onto $\ker W^*$). Moreover, we compute

$$ES = 4W^*PG_0PWW^*PW = 4W^*PG_0P(\mathbf{1} - G_0)PW = E - \frac{1}{2}E = \frac{1}{2}E,$$

and also $SE = \frac{1}{2}E$. Now we define

$$F = S - \frac{1}{2}E.$$

F is a projection,

$$F^2 = S^2 - \frac{1}{2}ES - \frac{1}{2}SE + \frac{1}{4}E^2 = (S - \frac{1}{4}E) - \frac{1}{4}E - \frac{1}{4}E + \frac{1}{4}E = F,$$

and we have the relation

$$\mathbf{1} - F - \Gamma F\Gamma = \mathbf{1} - (S - \frac{1}{2}E) - (\mathbf{1} - S - \frac{1}{2}E) = E.$$

Moreover, we have $F \perp \Gamma F \Gamma$ since

$$\begin{aligned} F\Gamma F\Gamma &= (S - \frac{1}{2}E)(\mathbf{1} - S - \frac{1}{2}E) = S - S^2 - \frac{1}{2}SE - \frac{1}{2}E + \frac{1}{2}ES + \frac{1}{4}E \\ &= \frac{1}{4}E - \frac{1}{4}E - \frac{1}{2}E + \frac{1}{4}E + \frac{1}{4}E = 0. \end{aligned}$$

For proving that F is a partial basis projection with Γ-codimension 1, it remains to be shown that E is a rank-one-projection. We do that by computing that its trace is one,

$$\begin{aligned} \mathrm{tr}(E) &= 4\,\mathrm{tr}(W^*PG_0PW) = 4\,\mathrm{tr}(PG_0PWW^*) = 4\,\mathrm{tr}(PG_0P(\mathbf{1} - G_0)) \\ &= 4\,\mathrm{tr}(G_0P) - 2\,\mathrm{tr}(PG_0) = 2\,\mathrm{tr}(PG_0) = 2\sum_{n=0}^{\infty}\langle g_n, PG_0g_n\rangle = 2\langle g_0, Pg_0\rangle = 1 \end{aligned}$$

where $\{g_n, n \in \mathbf{N}_0\}$ is any ONB of $\mathcal{K}$ containing g_0. Now the conditions for the application of Lemma 4.1 are fulfilled, therefore $\omega_P \circ \varrho_W$ decomposes into two pure states, giving rise to inequivalent representations. But since $\langle g_0, Pg_0\rangle = \frac{1}{2}$ and the normed vector g_0 spannes the kernel of W^*, we find $\ker W^* \cap P\mathcal{K} = \{0\}$ and therefore $(\mathcal{H}_P, \pi_P \circ \varrho_W, |\Omega_P\rangle)$ is a GNS representation of the state $\omega_P \circ \varrho_W$ by Lemma 2.4. It follows that $\pi_P \circ \varrho_W$ decomposes into two inequivalent representations, q.e.d.

Now we come to the general case that the corank of a Bogoliubov operator V is odd.

Theorem 4.3 *Let ω_P be a Fock state and let $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ be the corresponding Fock representation of the selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$. Further let $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ be a Bogoliubov operator with finite odd corank, i.e. $M_V = 2N + 1$, $N \in \mathbf{N}_0$, and let ϱ_V be the corresponding Bogoliubov endomorphism. Then the representation $\pi_P \circ \varrho_V$ decomposes into 2^{N+1} irreducible subrepresentations, namely we have*

$$\pi_P \circ \varrho_V = \left(\bigoplus_{j=1}^{2^N} \pi_+^{(j)}\right) \oplus \left(\bigoplus_{j=1}^{2^N} \pi_-^{(j)}\right). \tag{20}$$

Representations $\pi_\pm^{(j)}$ are not Fock representations. Moreover, for all $j, j' = 1, 2, \ldots, 2^N$, representations $\pi_+^{(j)}$ and $\pi_+^{(j')}$ are unitarily equivalent, also $\pi_-^{(j)}$ and $\pi_-^{(j')}$ are unitarily equivalent but representations $\pi_+^{(j)}$ and $\pi_-^{(j')}$ are inequivalent, and, since irreducible, disjoint.

Proof. We choose again a Γ-invariant ONB $\{e_n, n \in \mathbf{N}\}$ of $\mathcal{K}$, $e_n = \Gamma e_n$, $n \in \mathbf{N}$. Furthermore, we choose a Γ-invariant ONB $\{f_n, n = 0, 1, \ldots, 2N\}$ of $\ker V^*$, $f_n = \Gamma f_n$, $n = 0, 1, \ldots, 2N$. We define

$$f_{2N+n} = Ve_n, \qquad n = 1, 2, \ldots$$

Then $\{f_n, n \in \mathbf{N}_0\}$ is an ONB of $\mathcal{K}$, too, and we can write

$$V = \sum_{n=1}^{\infty} |f_{2N+n}\rangle\langle e_n|.$$

We introduce Bogoliubov operators $W_1 \in \mathcal{I}^1(\mathcal{K},\Gamma)$ and $W_2 \in \mathcal{I}^2(\mathcal{K},\Gamma)$,

$$W_1 = \sum_{n=1}^{\infty} |f_n\rangle\langle e_n|, \qquad W_2 = \sum_{n=0}^{\infty} |f_{n+2}\rangle\langle f_n|.$$

such that

$$V = W_2^N W_1, \qquad \varrho_V = \varrho_{W_2}^N \varrho_{W_1}.$$

By Theorem 3.2 it follows that

$$\pi_P \circ \varrho_V = \bigoplus_{j=1}^{2^N} \pi_{P_j} \circ \varrho_{W_1}$$

where π_{P_j}, $j = 1, 2, \ldots, 2^N$ are mutually equivalent irreducible (Fock) representations. Now we can use Lemma 4.2: Each $\pi_{P_j} \circ \varrho_{W_1}$ decomposes into two inequivalent irreducible (non-Fock) representations,

$$\pi_{P_j} \circ \varrho_{W_1} = \pi_+^{(j)} \oplus \pi_-^{(j)}.$$

Since the representations $\pi_{P_j} \circ \varrho_{W_1}$ are mutually equivalent, we can choose the $\pm$-sign such that $\pi_+^{(j)}$ and $\pi_+^{(j')}$ are equivalent, also $\pi_-^{(j)}$ and $\pi_-^{(j')}$, and that $\pi_+^{(j)}$ and $\pi_-^{(j')}$ are disjoint, $j, j' = 1, 2, \ldots, 2^N$, q.e.d.

We have seen that in the case $V \in \mathcal{I}^{2N+1}(\mathcal{K},\Gamma)$ the representation $\pi_P \circ \varrho_V$ is a direct sum of 2^{N+1} irreducible representations. Thus there must exist a set of 2^{N+1} disjoint projections in $\mathcal{B}(\mathcal{H}_P)$, commuting with $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K},\Gamma))$, which sum up to unity. By setting

$$g_0 = f_0, \qquad g_{\pm j} = \frac{1}{\sqrt{2}}(f_j \pm i f_{N+j}), \qquad j = 1, 2, \ldots, N,$$

we find an ONB $\{g_j, j = 0, \pm 1, \ldots, \pm N\}$ of $\ker V^*$ with the property $g_j = \Gamma g_{-j}$, $j = 0, 1, \ldots, N$. Let again $T_P(-1) \in \mathcal{B}(\mathcal{K})$ be a unitary operator which implements the automorphism α_{-1} of $\mathcal{C}(\mathcal{K},\Gamma)$, Eq. (6), in the Fock representation π_P. Since $\alpha_{-1}^2 = id$ the unitary $T_P(-1)$ can be choosen to be selfadjoint, i.e. $T_P(-1)^2 = \mathbf{1}$. We have

$$\pi_P(B(g_0))T_P(-1) = -T_P(-1)\pi_P(B(g_0)) \in \pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K},\Gamma))',$$

and by $B(g_0)^2 = \frac{1}{2}\mathbf{1}$ we find that

$$\Pi_0^{\pm} = \frac{1}{2}(\mathbf{1} \pm \sqrt{2} i \pi_P(B(g_0))T_P(-1))$$

are disjoint projections in $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K},\Gamma))'$. Also the projections

$$\Pi_j^{\pm} = \pi_P(B(g_{\pm j})^* B(g_{\pm j})), \qquad j = 1, 2, \ldots, N$$

lie in the commutant of $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K},\Gamma))$, one has $\Pi_j^+ + \Pi_j^- = \mathbf{1}$ for $j = 0, 1, 2, \ldots, N$, and all these projections commute mutually. Now we are able to construct 2^{N+1} projections

$$\Pi_{\epsilon_0,\epsilon_1,\ldots,\epsilon_N} = \Pi_0^{\epsilon_0}\Pi_1^{\epsilon_1}\cdots\Pi_N^{\epsilon_N}, \qquad \epsilon_j = \pm, \qquad j = 0, 1, \ldots, N$$

which have the desired properties.

Because the decomposition of the representation $\pi_P \circ \varrho_V$ into irreducible subrepresentations contains here two different equivalence classes we simply conclude that $\pi_P \circ \varrho_V$ cannot be unitarily equivalent to a multiple of π_P.

Corollary 4.4 *If $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ is a Bogoliubov operator with finite odd corank, i.e. $M_V = 2N + 1$, $N \in \mathbf{N}_0$, then representations π_P and $\pi_P \circ \varrho_V$ cannot be quasiequivalent for any Fock representation π_P of $\mathcal{C}(\mathcal{K}, \Gamma)$.*

We remark that this corollary agrees with Binnenhei's recent results on isometrical implementability of Bogoliubov endomorphisms [6].

5 Restriction to the Even Subalgebra

We now are interested in what happens when our representations of $\mathcal{C}(\mathcal{K}, \Gamma)$ become restricted to the even subalgebra $\mathcal{C}(\mathcal{K}, \Gamma)^+$ which is the algebra of fixpoints under the automorphism α_{-1} of Eq. (6),

$$\mathcal{C}(\mathcal{K}, \Gamma)^+ = \{x \in \mathcal{C}(\mathcal{K}, \Gamma) \,|\, \alpha_{-1}(x) = x\}.$$

We begin with a lemma which is taken from Araki's work [2].

Lemma 5.1 *Let $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ be a Fock representation of $\mathcal{C}(\mathcal{K}, \Gamma)$. In the restriction to the even subalgebra $\mathcal{C}(\mathcal{K}, \Gamma)^+$, the representation π_P splits into two irreducible subrepresentations,*

$$\pi_P|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \pi_P^+ \oplus \pi_P^-, \tag{21}$$

Representations π_P^+ and π_P^- are disjoint. The commutant of $\pi_P(\mathcal{C}(\mathcal{K}, \Gamma)^+)$ is generated by the α_{-1}-implementing $T_P(-1)$. The unitary $T_P(-1)$ can be choosen to be selfadjoint, i.e. $T_P(-1)^2 = \mathbf{1}$.

Remember that for a Bogoliubov operator V with $M_V = 2N$ a representation $\pi_P \circ \varrho_V$ splits into 2^N mutually equivalent Fock representations. Therefore $\pi_P \circ \varrho_V$, when restricted to the even subalgebra, decomposes into 2^{N+1} irreducibles where one has two different equivalence classes of 2^N mutually equivalent subrepresentations each. Now suppose $M_V = 2N + 1$. What happens with representations $\pi_\pm^{(j)}$ of Theorem 3.2, Eq. (20), in the restriction to the even subalgebra? They are associated to irreducible subrepresentations $\pi_{(F,\pm f_0)}$, Eq. (16), of representations π_S of $\mathcal{C}(\mathcal{K}, \Gamma)$, with corresponding quasifree states ω_S, S of the form (17). We will find that the situation becomes completely inverted: An irreducible Fock representation π_P splits, when restricted to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, into two inequivalent irreducibles. On the other hand, such a representation π_S splits already as representation of $\mathcal{C}(\mathcal{K}, \Gamma)$ into two inequivalent irreducible subrepresentatins, however, these subrepresentations, when restricted to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, remain irreducible but become equivalent.

Lemma 5.2 *The irreducible representation $\pi_{(F,f_0)}$, Eq. (16), of $\mathcal{C}(\mathcal{K}, \Gamma)$ remains irreducible when restricted to the even subalgebra $\mathcal{C}(\mathcal{K}, \Gamma)^+$. Moreover, in the restriction to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, representations $\pi_{(F,f_0)}$ and $\pi_{(F,-f_0)}$ become equivalent.*

Proof. For proving the irreducibilty of the restricted representation $\pi_{(F,f_0)}$, we assume an operator $A \in \mathcal{B}(\mathcal{H}_F)$ which commutes with every $\pi_{(F,f_0)}(x)$, $x \in \mathcal{C}(\mathcal{K}, \Gamma)^+$. Then we show that it follows $A = \lambda\mathbf{1}$, $\lambda \in \mathbf{C}$, immediately. Since $A \in \pi_{(F,f_0)}(\mathcal{C}(\mathcal{K}, \Gamma)^+)'$ the operator A commutes, in particular, with all representors of the subalgebra $\mathcal{C}((F + \Gamma F\Gamma)\mathcal{K}, \Gamma)^+$, i.e.

$$[A, \pi_{(F,f_0)}(x)] = 0, \qquad x \in \mathcal{C}((F + \Gamma F\Gamma)\mathcal{K}, \Gamma)^+.$$

But for $x \in \mathcal{C}((F + \Gamma F\Gamma)\mathcal{K}, \Gamma)^+$ we have $\pi_{(F,f_0)}(x) = \pi_F(x)$ since the first summand vanishes in Eq. (16). However, since π_F is a Fock representation of $\mathcal{C}((F + \Gamma F\Gamma)\mathcal{K}, \Gamma)$ we conclude that the commutant of $\pi_F(\mathcal{C}((F + \Gamma F\Gamma)\mathcal{K}, \Gamma)^+)$ is generated by $T_F(-1)$ by Lemma 5.1. This leads us to the ansatz

$$A = \lambda\mathbf{1} + \mu T_F(-1), \qquad \lambda, \mu \in \mathbf{C}.$$

Now choose a non-zero $f \in (F + \Gamma F \Gamma)\mathcal{K}$. We consider the representor of $B(f_0)B(f) \in \mathcal{C}(\mathcal{K},\Gamma)^+$,

$$\pi_{(F,f_0)}(B(f_0)B(f)) = \pi_{(F,f_0)}(B(f_0))\pi_{(F,f_0)}(B(f)) = \frac{1}{\sqrt{2}} T_F(-1)\pi_F(B(f)).$$

We compute

$$\begin{aligned}
[A, \pi_{(F,f_0)}(B(f_0)B(f))] &= [\mu T_F(-1), \frac{1}{\sqrt{2}} T_F(-1)\pi_F(B(f))] \\
&= \frac{\mu}{\sqrt{2}}(\pi_F(B(f)) - T_F(-1)\pi_F(B(f))T_F(-1)) \\
&= \sqrt{2}\mu\pi_F(B(f)).
\end{aligned}$$

But since $A \in \pi_{(F,f_0)}(\mathcal{C}(\mathcal{K},\Gamma)^+)'$ this commutator has to vanish. This implies $\mu = 0$ and therefore $A = \lambda\mathbf{1}$. It remains to be shown that $\pi_{(F,f_0)}$ and $\pi_{(F,-f_0)}$, when restricted to $\mathcal{C}(\mathcal{K},\Gamma)^+$, become equivalent. Now choose arbitrary $f_1, f_2 \in \mathcal{K}$. By Eq. (16) we compute

$$\begin{aligned}
&\pi_{(F,\pm f_0)}(B(f_1)B(f_2)) = \\
&\quad = \frac{1}{2}\langle f_0, f_1\rangle\langle f_0, f_2\rangle \pm \frac{1}{\sqrt{2}}\langle f_0, f_1\rangle T_F(-1)\pi_F(B(Ff_2 + \Gamma F\Gamma f_2)) \\
&\qquad \pm \frac{1}{\sqrt{2}}\langle f_0, f_2\rangle \pi_F(B(Ff_1 + \Gamma F\Gamma f_1))T_F(-1) + \pi_F(B(Ff_1 + \Gamma F\Gamma f_1))\pi_F(B(Ff_2 + \Gamma F\Gamma f_2)).
\end{aligned}$$

Since $T_F(-1)\pi_F(B(Ff_j + \Gamma F\Gamma f_j)) = -\pi_F(B(Ff_j + \Gamma F\Gamma f_j))T_F(-1)$, $j = 1, 2$, one sees easily

$$\pi_{(F,f_0)}(B(f_1)B(f_2)) = T_F(-1)\pi_{(F,-f_0)}(B(f_1)B(f_2))T_F(-1).$$

Now $\mathcal{C}(\mathcal{K},\Gamma)^+$ is generated by such elements $B(f_1)B(f_2)$. Therefore representations $\pi_{(F,f_0)}$ and $\pi_{(F,-f_0)}$ of $\mathcal{C}(\mathcal{K},\Gamma)^+$ are unitarily equivalent; the equivalence is realized by $T_F(-1)$, q.e.d.

Now we can apply these results to representations $\pi_P \circ \varrho_V$ where V is a Bogoliubov operator with finite corank. By applying Lemmata 5.1 and 5.2 to the subrepresentations of $\pi_P \circ \varrho_V$ according to Theorems 3.2 and 4.3 we conclude

Theorem 5.3 *Let ω_P be a Fock state and let $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ be the corresponding Fock representation of the selfdual CAR algebra $\mathcal{C}(\mathcal{K},\Gamma)$. Further let $V \in \mathcal{I}(\mathcal{K},\Gamma)$ be a Bogoliubov operator with finite corank, and let ϱ_V be the corresponding Bogoliubov endomorphism. If the corank of V is an even number, say $M_V = 2N$, $N \in \mathbf{N}_0$, then the representation $\pi_P \circ \varrho_V$, when restricted to the even subalgebra $\mathcal{C}(\mathcal{K},\Gamma)^+$, decomposes into 2^{N+1} irreducible subrepresentations, namely we have*

$$\pi_P \circ \varrho_V|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \left(\bigoplus_{j=1}^{2^N} \pi^+_{P_j}\right) \oplus \left(\bigoplus_{j=1}^{2^N} \pi^-_{P_j}\right). \tag{22}$$

For all $j, j' = 1, 2, \ldots, 2^N$, representations $\pi^+_{P_j}$ and $\pi^+_{P_{j'}}$ are unitarily equivalent, also $\pi^-_{P_j}$ and $\pi^-_{P_{j'}}$ are unitarily equivalent but representations $\pi^+_{P_j}$ and $\pi^-_{P_{j'}}$ are disjoint. On the other hand, if the corank of V is an odd number, say $M_V = 2N + 1$, $N \in \mathbf{N}_0$, then the representation $\pi \circ \varrho_V$, when restricted to $\mathcal{C}(\mathcal{K},\Gamma)^+$, decomposes into 2^{N+1} mutually equivalent irreducible subrepresentations.

6 Concluding Remarks

We have learned into how many irreducible subrepresentations a representation $\pi_P \circ \varrho_V$ of CAR splits in the case that the corank of the Bogoliubov operator V is finite. Moreover, we know something about

the equivalence classes of those subrepresentations. These results were gained by a suitable product decomposition of the Bogoliubov operator. Because one has to take into account a lot of distinctions, concerning the vanishing or non-vanishing of scalar products $\lambda_\pm$ in each step, we did not give explicit formulae for those subrepresentations for the case that $V \in \mathcal{I}^n(\mathcal{K}, \Gamma)$, $n > 2$. However, suppose such a Bogoliubov operator given explicitely, it can be seen from our proofs how one has to construct the subrepresentations (i.e. generating vectors (8),(12) and basis projections, Lemmata 2.4, 3.1) step by step. Therefore one always obtains an explicit construction. (We remark that in the case of equivalent subrepresentations, the decomposition into invariant subspaces is not unique.) Moreover, we have seen what happens with those representations when they become restricted to the even subalgebra. But what happens when the corank of the Bogoliubov operator V becomes infinite? Our product decomposition of V then becomes infinite; one cannot receive explicit formulae in this way, moreover, we cannot say anything about the equivalence classes of the subrepresentations of $\pi_P \circ \varrho_V$. However, because an ONB of $\ker V^*$ then becomes infinite it is possible to construct an unbounded number of disjoint projections in the commutant of $\pi_P \circ \varrho_V(\mathcal{C}(\mathcal{K}, \Gamma))$. Clearly, the representation $\pi_P \circ \varrho_V$ possesses a decomposition into infinitely many irreducibles.

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